

An algebraic approach to coarse graining

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Abstract

We propose that Kreimer's method of Feynman diagram renormalization via a Hopf algebra of rooted trees can be fruitfully employed in the analysis of block spin renormalization or coarse graining of inhomogeneous statistical systems. Examples of such systems include spin foam formulations of non-perturbative quantum gravity as well as lattice gauge and spin systems on irregular lattices and/or with spatially varying couplings. We study three examples which are Z_2 lattice gauge theory on irregular 2-dimensional lattices, Ising/Potts models with varying bond strengths and $(1 + 1)$ -dimensional spin foam models.

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1 Introduction

Spin foam models are a natural description of non-perturbative quantum gravity, both Lorentzian [1, 2] and euclidean [3, 4]. In these models, we need to sum over discrete spacetime histories and, on each history, perform a sum over labels which live in the space of representations of some compact or quantum group (usually $SU(2)$ or $SU_q(2)$). Thus, for each history, the sum to be performed defines a generalization of a spin system or a lattice gauge theory. At the same time, the sum over all histories may be thought of as a sum over Feynman diagrams, in which case the sums over labels replace the momentum integrals (see, for example, [4]).

The main problem in spin foam models is finding those theories which have a good continuum limit, in the sense that coarse-grained observables, averaged over many Planck lengths, describe a classical spacetime. While the combinatorial part of the problem of spin foam renormalization is similar to that of Feynman diagrams, the physical limits differ: in spin foam renormalization there is a physical cutoff, the Planck scale, and the interesting limit is that of a very large number of vertices, corresponding to an spacetime volume which is large in Planck units. Thus, the continuum limit of spin foam models should be formulated as a renormalization group problem.

The main motivation of the present work is to develop such a renormalization group approach for spin foam models. A particular spin foam can be thought of as a generalization of a spin system or a lattice gauge system, however, an inhomogeneous one, as it is an irregular 2-complex with varying bond strengths. Even if the bond strengths are taken to be the same initially, they will vary after a few block transformations of the irregular complex. A central ingredient in the solution of the renormalization problem for spin foams will then have to be a method to renormalize inhomogeneous systems. These are systems defined on irregular lattices, systems in which the bond strengths vary in space, or systems with both characteristics. It is more difficult to apply renormalization group techniques to such systems because there may be no bond strength which characterizes a particular scale. In this note we suggest that, for such systems, a generalization of the renormalization group exists, which is based on a Hopf algebra. The starting point is the recent discovery by Kreimer of a Hopf algebra which underlies quantum field theory renormalization [5], and has led to a new approach to perturbative renormalization which is currently under development [6].

We show here that a similar method applies to inhomogeneous spin systems and spin foams. As examples, we study the renormalization of the Z_2 lattice gauge theory on an inhomogeneous 2-dimensional lattice, and the 1-dimensional Ising model (2-dimensional Ising/Potts models are a straightforward generalization). We then give the renormalization operations on the partition function of a generic 1+1 spin foam.

The basic idea is that the (partitioned) spin systems are the elements of the algebra, and block transformations, which change a spin system to another by integrating

over subsets of spins, are operations on these elements. Among these elements are homogeneous systems. The standard renormalization group transformation changes a homogeneous system to another homogeneous system, and corresponds to the homogeneous coupling terms in a particular operation of the Hopf algebra. Under this new light, it is intriguing to consider the statement that the renormalization group is not a group because there is no inverse. A Hopf algebra is also not a group, but there is an operation which behaves as a generalized inverse, called the antipode. What we find here is that the renormalization group transformation of a spin system is closely related to the antipode. This is a non-perturbative analogue of Kreimer's results for renormalizable field theory, in which the antipode of a diagram is related to the counterterms necessary to subtract its divergencies.

In the next section, we define the Hopf algebra associated with the block transformations of the Z_2 lattice gauge theory on a 2-dimensional lattice. The renormalization of the theory via this algebra is illustrated for a simple example. In section 3, we give a simpler example of Hopf algebra block transformations, the 1-dimensional Ising model. Then, in section 4, we treat the formal partition function of spin foams in 1+1 dimensions. We define the Hopf algebra of partitioned spin foams and the general form of a coarse-graining operation. In the concluding section we indicate ways in which these results may be used. The properties of a Hopf algebra can be found in [7], and for rooted trees in [5].

2 Coarse-graining the Z_2 lattice gauge theory on a 2-dimensional lattice

In this section we study a simple example of the generalized renormalisation group transformation, carried out by the Hopf algebra of parenthesized Boltzmann weights.

We start by giving a 2-dimensional lattice Γ (which may be connected, or a set of disjoint lattices), with its edges labeled by Z_2 elements $q_i = \pm 1$. We call *plaquettes* the smallest loops in the lattice, namely, loops that contain no further loops. Each plaquette p is labeled by a coupling κ_p , a real number (figure 1).

A particular configuration (assignment of group elements and couplings) on the lattice Γ gives rise to a Boltzmann weight

$$w_\Gamma = \exp \left(\sum_{p \in \Gamma} \kappa_p \prod_{q_i \in p} q_i \right), \quad (1)$$

which is a function of the labels on the lattice. The product ranges over the labels on all the edges around the plaquette p . The temperature and Boltzmann's constant have been absorbed in the remaining parameters.

The partition function for the Z_2 lattice gauge theory is the sum of the weights

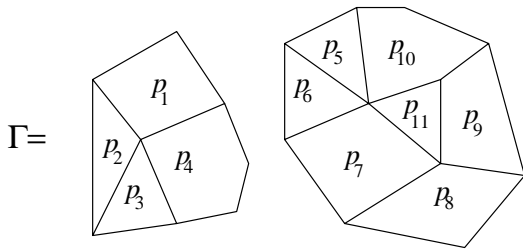


Figure 1: A 2-dimensional lattice with its plaquettes marked.

over all assignments of edge labels on Γ :

$$Z(\{\kappa_p\}) = \sum_{\{q_i\}} \exp \left(- \sum_{p \in \Gamma} \kappa_p \prod_{q_i \in p} q_i \right). \quad (2)$$

A particular case is when Γ is a regular lattice, for example, a square one. Then all the plaquettes are squares with sides of length a and all the couplings κ_p are the same. A renormalization group transformation on this system is a homogeneous coarse-graining of Γ to a new square lattice Γ' , with lattice spacing, say, $a' = 2a$. Such a transformation may be performed by following these steps: First, partition Γ into square sublattices γ , each with side $2a$. In each γ , sum over all labels on the internal edges, to obtain a new larger square plaquette p' labeled by a coupling $\kappa_{p'}$:

$$\kappa_{p'} = 2^4 \tanh^{-1} \left(\tanh^4 \kappa_p \right). \quad (3)$$

(For a derivation of this equation, see section 3.) Finally, we redefine the edges of the new plaquette so that p' is a square with new sides q'_1, q'_2, q'_3, q'_4 , each a product of the labels on two old sides. It contributes $\exp(-\kappa_{p'} q'_1 q'_2 q'_3 q'_4)$ to the partition function.

We repeat this procedure until we reach the desired new lattice spacing. The result is the renormalization group equation, the generalization of (3) to all the plaquettes on the final lattice. It has the form

$$\{\kappa'\} = R(\{\kappa\}), \quad (4)$$

which provides the couplings $\{\kappa'\}$ on the new lattice Γ' in terms of the couplings $\{\kappa\}$ on Γ .

Of course, summing over all the internal labels is only one possible coarse-graining scheme. Other schemes, exact or approximate (such as decimation or truncation), can be used as long as they respect the gauge invariance of the theory.

Our aim in this section is to generalise this coarse-graining procedure to inhomogeneous lattices in a way that gives rise to a Hopf algebra and an extension of the renormalisation group equation (4).

2.1 The Hopf algebra of partitioned Z_2 lattices and their weights

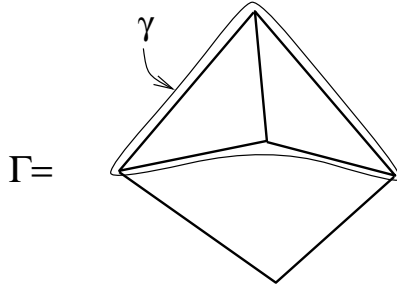
First we need to formalize the correspondence between Boltzmann weights and the underlying lattices. We call a lattice Γ *partitioned* when it is marked with an *allowed* partition into a set of sublattices. An allowed partition contains no overlapping sublattices. Namely, for any two sublattices γ_1, γ_2 in the partition, either $\gamma_1 \subset \gamma_2$, $\gamma_2 \subset \gamma_1$, or, $\gamma_1 \cap \gamma_2 = \emptyset$. A sublattice may be a set of disconnected non-overlapping sublattices.

Let us call V the space of labeled partitioned lattices over the real numbers. A general element in V is a sum of labeled partitioned lattices, with real coefficients. We can turn V into an algebra by defining *multiplication*, $m : V \otimes V \rightarrow V$, to be the disjoint union of two lattices:

$$\Gamma_1 \cdot \Gamma_2 = \Gamma_1 \cup \Gamma_2. \quad (5)$$

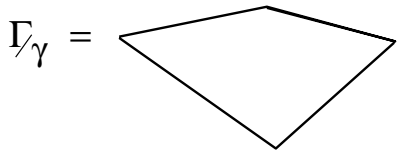
We have written $\Gamma_1 \cdot \Gamma_2$ for $m(\Gamma_1 \otimes \Gamma_2)$. The unit element is the empty lattice e . The *unit* operation, $\epsilon : R \rightarrow V$, turns a real number into a lattice by multiplying it with the empty lattice, $\epsilon(r) = re$, $r \in R$. The generators in V are the connected labeled partitioned lattices $\{\Gamma_c\}$.

We will now define the rest of the operations needed for V to be a Hopf algebra. First, let γ denote a proper sublattice of Γ , namely $\gamma \neq e$ and $\gamma \neq \Gamma$. We call the lattice that remains if we “cut out” γ from Γ , the *remainder*, and denote it by Γ/γ . That is, in the lattice



(6)

with the marked sublattice γ , the remainder is¹



(7)

¹This definition of the remainder is the simplest one when the weight on a lattice factorizes into weights on its sublattices in the partition and the renormalization scheme preserves this. Otherwise, the remainder should be defined to have the same external edges as the original. This is the case for the remainder in Section 4 for a spin foam, which is also the same as in [5, 6].

The *coproduct* in V is the operation $\Delta : V \rightarrow V \otimes V$ defined by

$$\Delta(\Gamma) = \Gamma \otimes e + e \otimes \Gamma + \sum_{\gamma} \gamma \otimes \Gamma/\gamma \quad (8)$$

$$\Delta(e) = e \otimes e \quad (9)$$

$$\Delta(\Gamma_1 \cdot \Gamma_2) = \Delta(\Gamma_1)\Delta(\Gamma_2). \quad (10)$$

The sum in (8) ranges over all sublattices γ *in the given partition* of Γ . Note that sublattices γ_p with no further sublattices are special. They satisfy

$$\Delta(\gamma_p) = \gamma_p \otimes e + e \otimes \gamma_p. \quad (11)$$

These are the *primitive elements* of the Hopf algebra. Plaquettes are always primitive.

The *counit* is an operation $\bar{\epsilon} : V \rightarrow R$ that annihilates every lattice except e :

$$\bar{\epsilon}(\Gamma) = \begin{cases} 0 & \text{for } \Gamma \neq e, \\ 1 & \text{for } \Gamma = e. \end{cases} \quad (12)$$

The counit and the coproduct satisfy $(\text{id} \otimes \epsilon)\Delta(\Gamma) = (\epsilon \otimes \text{id})\Delta(\Gamma) = \Gamma$. One can check that this coproduct is coassociative, i.e. it satisfies $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$. It is not cocommutative, namely, if we switch the order of the lattice pairs in each term in (8) we get an element of $V \otimes V$ different from $\Delta(\Gamma)$.

By adding the Δ and $\bar{\epsilon}$ operations to V , we have turned it into a bialgebra which is associative, coassociative and commutative. We now only need to define an *antipode* for V to be a Hopf algebra. An antipode is an operation $S : V \rightarrow V$, that satisfies

$$m(S \otimes \text{id})\Delta(\Gamma) = \epsilon\bar{\epsilon}(\Gamma) = \begin{cases} 0 & \text{for } \Gamma \neq e, \\ e & \text{for } \Gamma = e. \end{cases} \quad (13)$$

as well as $m(\text{id} \otimes S)\Delta(\Gamma) = \epsilon\bar{\epsilon}(\Gamma)$. We will need this later and so we name O the LHS of (13):

$$O(\Gamma) = m(S \otimes \text{id})\Delta(\Gamma) = \begin{cases} 0 & \text{for } \Gamma \neq e, \\ e & \text{for } \Gamma = e. \end{cases} \quad (14)$$

The antipode on the partitioned lattices is

$$S(\Gamma) = -\Gamma - \sum_{\gamma} S(\gamma)\Gamma/\gamma \quad (15)$$

$$S(\gamma_p) = -\gamma_p \quad (16)$$

$$S(e) = e \quad (17)$$

$$S(\Gamma_1 \cdot \Gamma_2) = S(\Gamma_1)S(\Gamma_2). \quad (18)$$

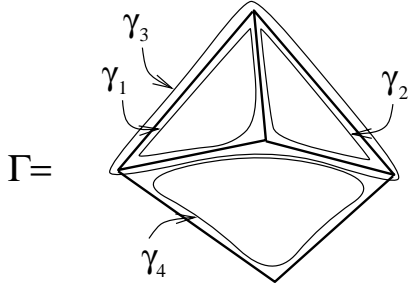
$S(\Gamma)$ is an iterative equation that stops when a primitive lattice γ_p is reached.

One can check that the operation (15) applied to a partitioned lattice Γ satisfies the (13). We thus have a Hopf algebra of partitioned labeled lattices.

That the lattices we use are partitioned is important because the partition dictates the order in which we will do the block spin transformation. However, the transformation is really an operation on the Boltzmann weight on a lattice. It is possible to encode the partitioning of a lattice in its Boltzmann weight by using *parenthesized* weights. This means writing the weight as a product the weights of each plaquette in the lattice, $w_\Gamma = \prod_p e^{\kappa_p} \prod q_i$, and marking by brackets the factors in the product that correspond to the sublattices in the given partition.

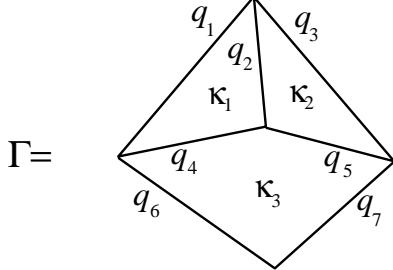
The rules are the following. The weight of a single sublattice γ is enclosed in a set of matching open/closing brackets (w_γ) . Two nested sublattices, $\gamma_1 \subset \gamma_2$ are put in nested brackets: $((w_{\gamma_1})w_{\gamma_2})$. Two disjoint sublattices are marked by a disjoint pair of brackets: $(w_{\gamma_1})(w_{\gamma_2})$. A single plaquette lives in a single set of brackets, with no nesting inside. Finally, if Γ is a connected lattice, the outermost left bracket matches the outermost right one².

For example, the parenthesized weight of the partitioned lattice



(19)

with labels



(20)

is

$$\begin{aligned} w_\Gamma &= (((w_{\gamma_1})(w_{\gamma_2}))(w_{\gamma_4})) \\ &= \left(\left(\left(e^{\kappa_{p_1}} \prod_{q_i \in p_1} q_i \right) \left(e^{\kappa_{p_2}} \prod_{q_i \in p_2} q_i \right) \right) \left(e^{\kappa_{p_3}} \prod_{q_i \in p_3} q_i \right) \right). \end{aligned} \quad (21)$$

From now on, when we write Γ we mean a (labeled) partitioned lattice, and by w_Γ we will mean a parenthesized weight.

²These rules are different than the parenthesized words of Kreimer in [5], since the weight of a lattice factorizes into weights of its sublattices, and we use an exact renormalization scheme which respects this property.

Therefore, to a partitioned labeled lattice corresponds a parenthesized weight, which is a real function of the labels on the lattice, of the form (1), marked with a bracket structure according to the above rules. In fact, we can think of a weight w as a map from V to the algebra of such parenthesized weights, which we will call W . All the operations we defined for V also apply to weights in W . Let us list them.

A general element in W is a sum of parenthesized weights with real coefficients. Multiplication in W is the product of the weights on two disjoint lattices:

$$w_{\Gamma_1} \cdot w_{\Gamma_2} = w_{\Gamma_1 \cup \Gamma_2}. \quad (22)$$

The unit element in W is 1, which we define to be the weight of the empty lattice: $w_e = 1$. The unit operation ϵ takes a real number r to rw_e . Again, there is a set of generating elements, the weights on connected lattices.

The coproduct is

$$\Delta(w_\Gamma) = w_\Gamma \otimes 1 + 1 \otimes w_\Gamma + \sum_{\gamma} w_\gamma \otimes w_{\Gamma/\gamma} \quad (23)$$

$$\Delta(1) = 1 \otimes 1 \quad (24)$$

$$\Delta(w_{\Gamma_1} w_{\Gamma_2}) = \Delta(w_{\Gamma_1}) \cdot \Delta(w_{\Gamma_2}). \quad (25)$$

$w_{\Gamma/\gamma}$ is the weight of the remainder, equal to $\frac{w_\Gamma}{w_\gamma}$.

Finally, the antipode is given by

$$S(w_\Gamma) = -w_\Gamma - \sum_{\gamma} S(w_\gamma) w_{\Gamma/\gamma} \quad (26)$$

$$S(w_{\gamma_p}) = -w_{\gamma_p} \quad (27)$$

$$S(1) = 1 \quad (28)$$

$$S(w_{\Gamma_1} w_{\Gamma_2}) = S(w_{\Gamma_1}) S(w_{\Gamma_2}). \quad (29)$$

It satisfies O given in (14).

2.2 Example

Let us now give an example of the coproduct and antipode operations on a partitioned lattice.

Consider the lattice (20), with the partition (19). The coproduct on this lattice produces all possible pairs of sublattices and remainders in the given partition. It is

$$\begin{aligned} \Delta(\Gamma) &= \Gamma \otimes e + e \otimes \Gamma + \gamma_1 \otimes \Gamma/\gamma_1 + \gamma_2 \otimes \Gamma/\gamma_2 \\ &\quad + \gamma_3 \otimes \Gamma/\gamma_3 + \gamma_4 \otimes \Gamma/\gamma_4 \\ &= \Gamma \otimes e + e \otimes \Gamma + \gamma_1 \otimes \Gamma/\gamma_1 + \gamma_2 \otimes \Gamma/\gamma_2 + \gamma_3 \otimes \gamma_4 + \gamma_4 \otimes \gamma_3. \end{aligned} \quad (30)$$

Next, we calculate the antipode. The lattices γ_1, γ_2 and γ_4 are primitive. For γ_3 , we have

$$\begin{aligned} S(\gamma_3) &= -\gamma_3 - S(\gamma_1)\gamma_3/\gamma_1 - S(\gamma_2)\gamma_3/\gamma_2 \\ &= -\gamma_3 + 2\gamma_1\gamma_2. \end{aligned} \quad (31)$$

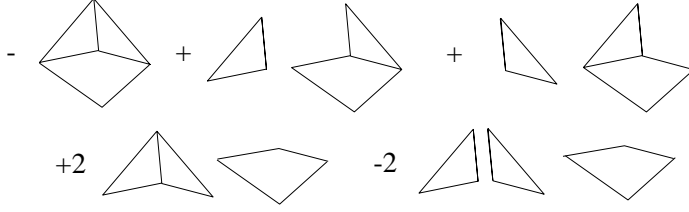
We plug this in

$$S(\Gamma) = -\Gamma - \sum_{i=1,\dots,4} S(\gamma_i)\Gamma/\gamma_i \quad (32)$$

and get

$$\begin{aligned} S(\Gamma) &= -\Gamma + \gamma_1\Gamma/\gamma_1 + \gamma_2\Gamma/\gamma_2 + \gamma_3\Gamma/\gamma_3 \\ &\quad - 2\gamma_1\gamma_2\Gamma/\gamma_3 + \gamma_4\Gamma/\gamma_4. \\ &= -\Gamma + \gamma_1\Gamma/\gamma_1 + \gamma_2\Gamma/\gamma_2 - 2\gamma_1\gamma_2\gamma_4 + 2\gamma_3\gamma_4. \end{aligned} \quad (33)$$

namely, the lattice



$$- \text{[tetrahedron]} + \text{[triangle]} \text{[quadrilateral]} + \text{[triangle]} \text{[quadrilateral]} + \text{[triangle]} \text{[quadrilateral]} + 2 \text{[triangle]} \text{[quadrilateral]} - 2 \text{[triangle]} \text{[quadrilateral]} \quad (34)$$

We can use (30) and (33) to check that $O(\Gamma) = 0$.

We can carry out the same calculations on the parenthesized weight w_Γ in (21) for this lattice. The result is exactly the same as (30) and (33), but we will write it out to indicate how parenthesized weights should be manipulated. For the coproduct, we have

$$\begin{aligned} \Delta(w_\Gamma) &= w_\Gamma \otimes 1 + 1 \otimes w_\Gamma + (w_{\gamma_1}) \otimes ((w_{\gamma_2})(w_{\gamma_4})) + (w_{\gamma_2}) \otimes ((w_{\gamma_1})(w_{\gamma_4})) \\ &\quad + ((w_{\gamma_1})(w_{\gamma_2})) \otimes (w_{\gamma_4}) + (w_{\gamma_4}) \otimes ((w_{\gamma_1})(w_{\gamma_2})). \end{aligned} \quad (35)$$

The antipode is

$$\begin{aligned} S(w_\Gamma) &= -w_\Gamma + (w_{\gamma_1})((w_{\gamma_2})(w_{\gamma_4})) + (w_{\gamma_2})((w_{\gamma_1})(w_{\gamma_4})) \\ &\quad + 2((w_{\gamma_1})(w_{\gamma_2}))(w_{\gamma_4}) - 2(w_{\gamma_1})(w_{\gamma_2})(w_{\gamma_4}). \end{aligned} \quad (36)$$

2.3 The shrinking “antipode”

We can perform a renormalization group operation on a lattice by summing over possible values on some, or all, edges internal in the lattice, and so shrinking the lattice down to one with a smaller number of plaquettes, carrying effective couplings. The partition function on the new lattice is a different function, on a different set of labels, than the original one. When new and old couplings obey the renormalization group equation, the value of the effective partition function is equal to the value of the original one.

We will reproduce this by using the operations of the Hopf algebra we defined and get the correct effective couplings by using a modified version of the antipode S of eq. 15. We will first give a general form of this operation, and then apply it to our Z_2 example.

First, we define an operation R which: 1) when applied to a lattice Γ , it produces an effective lattice $R(\Gamma)$, *with the same external edges and their labels* as Γ . $R(\Gamma)$ may also be a sum of lattices with the same external edges and labels as the original one. 2) when applied to a weight w_Γ on that lattice it produces a new weight $R(w_\Gamma)$ on $R(\Gamma)$.

We want R to be a renormalization operation, which means that we want an equivalence relation

$$R(\Gamma) \sim \Gamma, \quad (37)$$

$$R(w_\Gamma) \sim w_\Gamma, \quad (38)$$

which means that the two lattices are equivalent under renormalization.

Exactly what the relationship between the two weights depends on the chosen coarse-graining scheme. It is straightforward to state if R is an *exact* scheme. Then the partition function $Z_R(\Gamma)$ with weight $R(w_\Gamma)$ evaluates to the same number as the partition function $Z(\Gamma)$ with weight w_Γ , for all w_Γ in the theory. This will be the case, for example, if we define R to erase all internal edges on the lattice,

$$R(\Gamma) = \partial\Gamma, \quad (39)$$

by summing over all labels on the erased edges:

$$R(w_\Gamma) = \sum_{\{q_i \in \bar{\Gamma}\}} w_\Gamma. \quad (40)$$

($\bar{\Gamma}$ is the interior of Γ , as before). We will do a calculation of this kind of R on a Z_2 lattice in the following subsection.

In an *approximate* renormalization scheme, we expect that $R(w_\Gamma) \sim w_\Gamma$ if $Z(\Gamma) = Z_R(\Gamma) + Z_R^c(\Gamma)$, where $Z_R^c(\Gamma)$ is the correction terms, which should be appropriately small. For example, $R(w_\Gamma)$ may be truncation, or extraction of a pole term from w_Γ . In a decimation scheme, R will be an operation that chooses certain edges of Γ to be the edges of the new lattice, with the original edge labels, and throws away the rest.

For approximate R , we should note the following: $R(w_{\Gamma_1}) \sim w_{\Gamma_1}$ does *not* imply $R(w_{\Gamma_1})w_{\Gamma_2} \sim w_{\Gamma_1}w_{\Gamma_2}$. However, motivated by [5, 6], we will require that R on two lattices $\Gamma_1 = \prod_i \Gamma_1^i$ and $\Gamma_2 = \prod_j \Gamma_2^j$ satisfies

$$R\left(\prod_i R(\Gamma_1^i) \prod_j \Gamma_2^j\right) = \prod_i R(\Gamma_1^i) \prod_j R(\Gamma_2^j). \quad (41)$$

Now define a shrinking operation S_R as

$$S_R(\Gamma) = -R(\Gamma) - R\left(\sum_\gamma S_R(\gamma)\Gamma/\gamma\right) \quad (42)$$

on lattices, and

$$S_R(w_\Gamma) = -R(w_\Gamma) - R\left(\sum_{\gamma} S_R(w_\gamma) w_{\Gamma/\gamma}\right) \quad (43)$$

on weights. This is a modification of the antipode (15). The sum ranges over all proper sublattices in the given partition of Γ , as before, and stops when a primitive lattice is reached:

$$S_R(\gamma_p) = -\gamma_p. \quad (44)$$

We will call S_R the *shrinking antipode* (and sometimes refer to S as the “straight” antipode).

The equivalent of O can be written down for S_R , both for lattices and for weights

$$O_R = m(S_R \otimes \text{id})\Delta. \quad (45)$$

If O_R evaluates to the same right hand side as O , namely 0 for all $\Gamma \neq e$ and 0 for all $w_\Gamma \neq 1$, then we will have a Hopf algebra for the renormalization group. By this we mean that O_R is *equivalent* to 0 under the chosen R .

For the weights in the following two examples, it is the case that O_R is equivalent to 0 under the chosen renormalization map R . First, we will give the example of an exact block transformation on a Z_2 lattice, followed by an alternative definition for S_R , which applies to some coarse-graining schemes and is simpler. Next, in section 3, we give a simpler example than Z_2 , the 1-dimensional Ising model.

2.4 Shrinking example

In this example, we block transform the lattice (20) with partition (19).

We will use the R that eliminates all edges in the interior of the lattice:

$$R(\gamma) = \partial\gamma. \quad (46)$$

by summing over the labels on all the internal edges of γ :

$$R(w_\gamma) = \sum_{q_i \in \tilde{\gamma} = \pm 1} w_\gamma. \quad (47)$$

The sublattices $\gamma_1, \gamma_2, \gamma_4$ are primitive. For γ_3 , we calculate (42)

$$S_R(\gamma_3) = -\partial\gamma_3 + 2\gamma_1\gamma_2, \quad (48)$$

which we plug in (42) for Γ , and find

$$S_R(\Gamma) = -\partial\Gamma + \gamma_1\partial(\Gamma/\gamma_1) + \gamma_2\partial(\Gamma/\gamma_2) + 2\partial(\gamma_3)\gamma_4 - 2\gamma_1\gamma_2\gamma_4. \quad (49)$$

namely, the lattice

$$\begin{aligned}
& - \text{[square]} + \text{[triangle]} \text{[pentagon]} + \text{[triangle]} \text{[pentagon]} \\
& + 2 \text{[triangle]} \text{[pentagon]} - 2 \text{[triangle]} \text{[pentagon]}
\end{aligned} \tag{50}$$

The corresponding expression for weights needs:

$$\begin{aligned}
w'_{\gamma_3} &:= R(w_{\gamma_3}) = e^{\kappa'_3 q_1 q_3 q_5 q_4}, \\
\text{with } \kappa'_3 &= \tanh^{-1}(\tanh \kappa_1 \tanh \kappa_2),
\end{aligned} \tag{51}$$

$$\begin{aligned}
w'_{\Gamma/\gamma_1} &:= R(w_{\Gamma/\gamma_1}) = e^{\kappa'_2 q_2 q_3 q_7 q_6 q_4}, \\
\text{with } \kappa'_2 &= \tanh^{-1}(\tanh \kappa_2 \tanh \kappa_3),
\end{aligned} \tag{52}$$

$$\begin{aligned}
w'_{\Gamma/\gamma_2} &:= R(w_{\Gamma/\gamma_2}) = e^{\kappa'_1 q_1 q_2 q_5 q_7 q_6}, \\
\text{with } \kappa'_1 &= \tanh^{-1}(\tanh \kappa_1 \tanh \kappa_3),
\end{aligned} \tag{53}$$

$$\begin{aligned}
w'_\Gamma &:= R(w_\Gamma) = e^{\kappa' q_1 q_3 q_7 q_6}, \\
\text{with } \kappa' &= \tanh^{-1}(\tanh \kappa_1 \tanh \kappa_2 \tanh \kappa_3),
\end{aligned} \tag{54}$$

(we have used the method described in the footnote in Section 3) and results in

$$S_R(\Gamma) = -w'_\Gamma + (w_{\gamma_1})(w'_{\Gamma/\gamma_1}) + (w_{\gamma_2})(w'_{\Gamma/\gamma_2}) + 2(w'_{\gamma_3})(w_{\gamma_4}) - 2(w_{\gamma_1})(w_{\gamma_2})(w_{\gamma_4}). \tag{55}$$

Substituting (49) and (30) in (45), we find

$$O_R(\Gamma) = \Gamma - \partial\Gamma + \gamma_1 \partial(\Gamma/\gamma_1) - \gamma_1 \Gamma/\gamma_1 + \gamma_2 \partial(\Gamma/\gamma_2) - \gamma_2 \Gamma/\gamma_2 - \gamma_3 \gamma_4 + \partial(\gamma_3) \gamma_4, \tag{56}$$

namely the lattice

$$\begin{aligned}
& \text{[tetrahedron]} - \text{[square]} + \text{[triangle]} \text{[pentagon]} - \text{[triangle]} \text{[pentagon]} \\
& + \text{[triangle]} \text{[pentagon]} - \text{[triangle]} \text{[pentagon]} - \text{[triangle]} \text{[pentagon]} + \text{[triangle]} \text{[pentagon]}
\end{aligned} \tag{57}$$

This is zero under the equivalence relation $\gamma \sim R(\gamma) = \partial\gamma$ in (38) and (46). The corresponding antipode for weights also gives $O_R(w_\Gamma) = 0$ under $w_\gamma \sim R(w_\gamma)$ and (47).

2.5 An alternative definition of S_R on the Z_2 lattice

Keeping R as above, we will now give a different definition of S_R , that can be used in an exact renormalization scheme like (46), (47). We will define

$$S'_R(\Gamma) = -R(\Gamma) - \sum_{\gamma} S'_R(\gamma)\Gamma/\gamma, \quad (58)$$

with the same expression also applying to the corresponding weights. It is different than S_R since it is missing the overall R operation. Since, for our renormalization scheme (46), (47), $R(\Gamma_1 \cdot \Gamma_2) = R(\Gamma_1)R(\Gamma_2)$, S'_R differs from S_R in the contributions of the remainders, which are not shrunk.

On our example, this gives

$$S'_R(\Gamma) = -\partial\Gamma + \gamma_1\Gamma/\gamma_1 + \gamma_2\Gamma/\gamma_2 + \partial(\gamma_3)\Gamma/\gamma_3 - 2\gamma_1\gamma_2\gamma_4 + \gamma_4\Gamma/\gamma_4, \quad (59)$$

which is the lattice

$$- \text{square} + \text{triangle-square} + \text{triangle-square} + \text{triangle-square} - 2 \text{triangle-square} + \text{triangle-square} \quad (60)$$

Plugging (30) and (59) in O_R , we get

$$O_R(\Gamma) = \Gamma - R(\Gamma). \quad (61)$$

Again this is zero under \sim .

This definition of the shrinking antipode is attractive if we wish to use the second term in (59) as an iterative equation, which calculates the effective weights on the shrunk lattice in terms of the weights on its sublattices.

3 The Hopf algebra renormalization of the 1-d Ising model

We now illustrate the method on a simple 1-dimensional example. Consider a finite Ising chain Γ . This is a line, or a circle, with N spins attached to it. The coupling between spins s_i and s_j is κ_{ij} when i, j are adjacent sites and zero otherwise. A Boltzmann weight for Γ is

$$w_\Gamma = \exp \left(\sum_{\langle i, j \rangle} \kappa_{ij} s_i s_j \right), \quad (62)$$

where $\langle i, j \rangle$ means that i and j are adjacent sites in the chain. As before, we absorb β in the other parameters. Thus, the partition function for the system is

$$Z(\Gamma) = \sum_{\{s_i\}} \exp\left(- \sum_{\langle i, j \rangle} \kappa_{ij} s_i s_j\right). \quad (63)$$

Clearly, partitioned Ising chains give rise to a Hopf algebra, as they are a special case of the 2-dimensional lattices we already analyzed. So do their Boltzmann weights. Given an allowed partition of Γ , a subchain γ will either be a sequence of n spins, with weight $w_\gamma = \prod_{i=k, \dots, k+n} e^{\kappa_{ii+1} s_i s_{i+1}}$, or a disjoint union of such subchains, $\gamma_1 \cup \gamma_2$, with weight $w_{\gamma_1} w_{\gamma_2}$. The remainder Γ/γ is the subchain of Γ that contains all spins except those internal in γ and has weight $w_{\Gamma/\gamma} = w_\Gamma / w_\gamma$.

The coproduct (8), again produces all possible pairs of subchains and remainders in the given partition of Γ . We can easily identify the primitive elements. There is only one type of primitive chain: a pair of adjacent spins. A primitive weight then has the form $\gamma_p = e^{\kappa_{ij} s_i s_j}$.

To block transform Γ using the Hopf algebra, we first define the R operation on such subchains as

$$R(\gamma) = \partial\gamma, \quad (64)$$

$$R(w_\gamma) = \sum_{\text{Internal spins of } \gamma=\pm 1} w_\gamma. \quad (65)$$

For $\gamma = \gamma_1 \cup \gamma_2$, $R(\gamma) = R(\gamma_1)R(\gamma_2)$ and, on a primitive diagram, $R(\gamma_p) = \gamma_p$.

The details of the shrinking operation are best illustrated with examples. The point can be made with an Ising model with 3 spins. For larger chains, nothing new happens, except that the combinatorics produce more terms at each step.

Consider then the chain

$$\Gamma = \begin{array}{c} \gamma_1 \quad \gamma_2 \\ \bullet \text{---} \bullet \text{---} \bullet \\ s_1 \quad s_2 \quad s_3 \end{array}, \quad (66)$$

with weight $w_\Gamma = e^{\kappa_{12} s_1 s_2} e^{\kappa_{23} s_2 s_3}$, partitioned into subchains γ_1, γ_2 , with $w_{\gamma_1} = e^{\kappa_{12} s_1 s_2}$ and $w_{\gamma_2} = e^{\kappa_{23} s_2 s_3}$. They are both primitive.

The coproduct on Γ is

$$\Delta[\Gamma] = \Gamma \otimes e + e \otimes \Gamma + \gamma_1 \otimes \gamma_2 + \gamma_2 \otimes \gamma_1. \quad (67)$$

The “straight” antipode S of eq. (15), gives

$$S(\Gamma) = -\Gamma + 2\gamma_1\gamma_2. \quad (68)$$

One can check that $O(\Gamma) = 0$.

The shrinking antipode S'_R (eq. (59)) on w_Γ , gives

$$\begin{aligned} S'_R(w_\Gamma) &= -R(w_\Gamma) + R(w_{\gamma_1}) w_{\gamma_2} + R(w_{\gamma_2}) w_{\gamma_1} \\ &= w_{\Gamma'} + 2w_{\gamma_1} w_{\gamma_2}, \end{aligned} \quad (69)$$

where $w_{\Gamma'} = R(w_\Gamma)$, which, using eq.(65) is³

$$w_{\Gamma'} = e^{\kappa'} s_1 s_3, \quad \text{with } \kappa' = \tanh^{-1}(\tanh \kappa_1 \tanh \kappa_2). \quad (72)$$

Therefore, $O_R(w_\Gamma) = -w_{\Gamma'} + w_\Gamma$. That is, the fully blocked chain Γ' is:

$$w_{\Gamma'} = w_\Gamma - O_R(w_\Gamma). \quad (73)$$

For a larger Ising chain, either $w_{\Gamma'} = e^{K s_1 s_N}$, if Γ is an open chain with external spins s_1 and s_N (and K is the overall effective coupling, the generalization of (72) to N spins), or $w_{\Gamma'} = e^{K s_i^2}$ for some spin $s_i \in \Gamma$, if Γ is a closed chain. This can be thought of as the “fully block transformed” Γ , or the evaluation of $w(\Gamma)$.

An Ising/Potts model in two dimensions works in the same way.

It is important to note the following. In this model, as well as in the Z_2 case, eq.(73) is redundant in the calculation of the fully block transformed chain Γ' , since we have already calculated it as the term $R(w_\Gamma)$ in $S_R(w_\Gamma)$. However, this is a special property of these models and the exact renormalization scheme R that we employed. It will not be the case, for example, in spin foam models, where we do need to calculate O_R . We discuss this in the next section.

4 Basics of the Hopf algebra renormalization of a 1+1 spin foam

We now show that a similar Hopf algebra is defined on 1+1 spin foams and can be used in the renormalization of spin foam models. In 1+1 dimensions, a spin foam Γ is a 2-dimensional lattice, with vertices v and faces f . The faces are labeled by

³There are several ways to derive this formula and I will quickly outline one:
In $w(\Gamma)$, expand each factor using

$$e^{\kappa s_i s_j} = \cosh \kappa (1 + x s_i s_j), \quad (70)$$

with x given by $x = \tanh(\kappa)$. Then, keep s_1 and s_3 fixed and sum over $s_2 = \pm 1$. Only terms with even powers of internal spins survive, and we are left with

$$2 \cosh \kappa_1 \cosh \kappa_2 (1 + (x_1 x_2) s_1 s_3). \quad (71)$$

We ignore the factor 2 as it does not affect the calculation of any expectation values, and note that this is a nearest-neighbour interaction $e^{\kappa' s_1 s_3}$ if we redefine the coupling to be κ' given by equation (72).

unitary irreducible representations a_f of a Lie group G . $\dim a_f$ is the dimension of the representation. Each vertex v is labeled by an amplitude $A(v)$, a function of the labels on the faces adjacent to that vertex. Particular choices of the group and these functions give rise to specific spin foam models [3]. Also, the faces of Γ may be labeled by integers representing geometric properties such as lengths or matter[2].

The partition function for a spin foam has the form

$$Z = \sum_{\Gamma} N(\Gamma) \sum_{\substack{\text{Labelings} \\ \text{on } \Gamma}} \prod_{f \in \Gamma} \dim a_f \prod_{v \in \Gamma} A_v. \quad (74)$$

where the first sum ranges over all spin foams that extrapolate between fixed initial and final spin networks. We will treat the second sum in Z as a generalization of a lattice gauge theory and list the basic features of its renormalization by the Hopf algebra method.

We will call “subfoam” a proper sublattice γ of a spin foam Γ (one that is not empty and not Γ itself). As in the case of the Z_2 lattice gauge theory, we will work with *partitioned* spin foams, namely spin foams which have been marked by a partition into subfoams in which no two subfoams overlap.

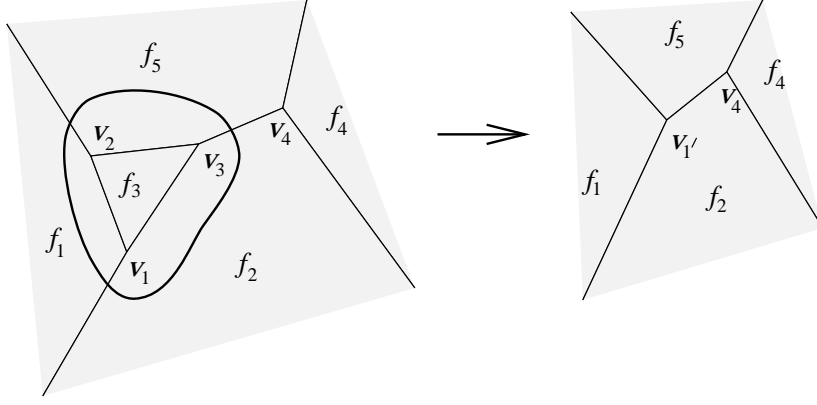
Let A be the collection of partitioned 1+1 spin foams. Since each spin foam can be multiplied by a complex number, we will think of A as an algebra over the complexes. *Multiplication* is the disjoint union of two spin foams: $\Gamma_1 \cdot \Gamma_2 = \Gamma_1 \cup \Gamma_2$. Denoting the empty spin foam by e , $\Gamma \cdot e = e \cdot \Gamma$, for every spin foam Γ . The *unit* operation is the map $\epsilon : \mathbf{C} \rightarrow A$, which for some complex number c gives $\epsilon(c) = ce$. At least in 1+1 dimensions, A is a commutative algebra since the order of two disjoint spin foams does not matter.

Two further operations that are natural for spin foams turn A into a coalgebra. First, the *counit* annihilates all spin foams, except the empty one:

$$\bar{\epsilon}(\Gamma) = \begin{cases} 0 & \text{for } \Gamma \neq e, \\ 1 & \text{for } \Gamma = e. \end{cases} \quad (75)$$

The remainder Γ/γ of a subfoam γ is the subfoam obtained by shrinking γ to a

point in Γ :



(76)

Note that this is different than the remainder we used in section 2, where we “cut out” γ . Rather, it is the same as Kreimer’s remainder in [5].

The *coproduct* splits a spin foam into a sum of all its possible subfoams, paired to their remainders:

$$\Delta(\Gamma) = \Gamma \otimes e + e \otimes \Gamma + \sum_{\gamma} \gamma \otimes \Gamma/\gamma. \quad (77)$$

As before, γ in the above sum ranges over all subfoams in the given partition. For a spin foam Γ with no subfoams, i.e. a *primitive spin foam*, we have

$$\Delta(\Gamma) = \Gamma \otimes e + e \otimes \Gamma. \quad (78)$$

If there is a restriction in the valence of the spin foam vertices in a given spin foam model, then there is a finite set of primitive spin foams⁴.

The straight antipode on spin foam diagrams is S as given in eq.(15), but with the remainder defined above.

All of the above operations on the spin foam complexes have their counterparts for the spin foam weights, as in the lattice gauge theory we have already studied in detail. Thus, in the above example (76), the weight for Γ is

$$w_{\Gamma} = \prod_{i=1,\dots,5} \dim a_{f_i} \prod_{k=1,\dots,4} A_{v_k}, \quad (79)$$

while the marked subfoam has weight

$$w_{\gamma} = \prod_{i=1,2,3,5} \dim a_{f_i} \prod_{k=1,2,3} A_{v_k}. \quad (80)$$

For the shrinking antipode S_R , we need to define a renormalization recipe R that provides effective vertices in terms of the original ones. We have no explicit

⁴These primitive foams can be compared to Feynman diagrams with no subdivergences. For a given theory, for example ϕ^4 , we can list the diagrams with no subdivergences, of which there is a finite number.

renormalization scheme to suggest in this paper, so we will simply note three basic things. One is that a possible R operation is the recoupling moves. For example, it is possible to shrink the subfoam in the above example using a 3-to-1 move.

Second, if the theory is triangulation invariant the effective vertices contain no information about the ones that we have shrunk. This makes the algebra trivial.

Finally, we note that the equivalence relation $R(\Gamma) \sim \Gamma$ modifies the summation over all interpolating spin foams in the partition function. In general, there is an infinite number of such foams, which makes it difficult to handle this sum in any context other than triangulation invariance or renormalized spin foams. However, we can split the first sum in (74) into sums over spin foams that are equivalent under renormalization. Developing a particular scheme to understand the renormalization group flow would then enable us to calculate the partition function.

5 Discussion

We have expressed block spin transformations of spin systems and spin foam as an equivalence relation and a modification of the antipode of a Hopf algebra, originally used by Kreimer for the perturbative renormalization of quantum field theory.

As a method to carry out the renormalization group transformation of spin systems, this is promising especially on inhomogeneous lattices, as it can efficiently keep track of the combinatorial part of the problem. The antipode of the algebra, which produces the generalized renormalization group equation, is an iterative equation and thus ready to be implemented numerically. (Broadhurst and Kreimer easily calculated 4d Yukawa theory to 30 loops using the perturbative form of this algebra [8].)

However, the summations involved in a single block transformation may still be formidable. In sections 2 and 3, we discussed exact renormalization schemes of spin systems. In fact, the power of the algebra is with approximate schemes, such as truncations of the Boltzmann weights. This is also the case for spin foams. Such schemes will be studied in future work.

We have given the operations of the algebra on the generic spin foam partition function. The equivalence relation defined via this algebra defines equivalence classes of spin foams, corresponding to the same effective vertex, and we argued that this may be used to reduce the number of spin foams to be summed over in the partition function.

As we discussed in 2.3, if the condition O_R is satisfied for every weight in the algebra, then the shrinking antipode is a genuine antipode. This would imply that the renormalization group equation can be embedded in this Hopf algebra. For the choices of weights in the examples we studied, this is satisfied. Whether this is generally the case for spin systems or gauge systems is left for further work.

We should note that our discussion applies to euclidean spin foams, as we have not paid attention to the orientation of the edges of the spin foam. However, we expect

that the construction can also be applied to the causal spin foams [1]. Also, although we have given the operations on a 1+1 spin foam, there should be no obstruction to an analogue in higher dimensions. The only necessary ingredient in this Hopf algebra is the rooted tree structure (the partitioned lattices), which of course exists in higher dimensions.

To determine whether the method here is a useful tool in spin foam renormalization, it is of course necessary to apply it to specific spin foam models. As we said above, topological state sum models appear unsuitable as the R operation is trivial. A good candidate is the Ambjorn-Loll-Anagnostopoulos Lorentzian gravity model in 2 dimensions, as it has a transfer matrix formulation and its continuum limit is known.

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References

- [1] Fotini Markopoulou, Dual formulation of spin network evolution, gr-qc/9704013.
 Fotini Markopoulou, Lee Smolin, Causal evolution of spin networks, Nucl.Phys. B508 (1997) 409.
 Fotini Markopoulou, Lee Smolin, Quantum geometry with intrinsic local causality, Phys.Rev. D58 (1998) 084032.
 Sameer Gupta, Causality in Spin Foam Models, Phys.Rev. D61 (2000) 064014.
- [2] Ambjorn J and Loll R, 1998, Non-perturbative Lorentzian Quantum Gravity, Causality and Topology Change, Nucl Phys B536 407.
 Ambjorn J, Nielsen JL, Rolf J and Loll R, Euclidean and Lorentzian Quantum Gravity - Lessons from Two Dimensions, Chaos Solitons Fractals 10 (1999) 177-195.
 J. Ambjorn, K.N. Anagnostopoulos, R. Loll, A new perspective on matter coupling in 2d quantum gravity, Phys.Rev. D60 (1999) 104035.
- [3] Michael P. Reisenberger, A lattice worldsheet sum for 4-d Euclidean general relativity, gr-qc/9711052.
 Michael P Reisenberger, Carlo Rovelli, "Sum over Surfaces" form of Loop Quantum Gravity, Phys.Rev. D56 (1997) 3490 .
 John W. Barrett, Louis Crane, Relativistic spin networks and quantum gravity, J.Math.Phys. 39 (1998) 3296 .
 John C. Baez, Spin Foam Models, Class.Quant.Grav. 15 (1998) 1827 .
 R. De Pietri, L.Freidel, so(4) Plebanski Action and Relativistic Spin Foam Model,

- Class.Quant.Grav. 16 (1999) 2187.
 R. De Pietri, Canonical “Loop” Quantum Gravity and Spin Foam Models, gr-qc/9903076.
 Junichi Iwasaki, A surface theoretic model of quantum gravity, gr-qc/9903112 .
 John C. Baez, An Introduction to Spin Foam Models of Quantum Gravity and BF Theory, gr-qc/9905087 .
 R. De Pietri, L. Freidel, K. Krasnov, C. Rovelli, Barrett-Crane model from a Boulatov-Ooguri field theory over a homogeneous space, Nucl.Phys. B574 (2000) 785-806.
- [4] J Baez, Spin Foam Perturbation Theory, gr-qc/9910050.
 Laurent Freidel, Kirill Krasnov, Simple Spin Networks as Feynman Graphs, J.Math.Phys. 41 (2000) 1681-1690.
 Michael P. Reisenberger, Carlo Rovelli, Spin foams as Feynman diagrams, gr-qc/0002083.
- [5] Dirk Kreimer, On the Hopf algebra structure of perturbative quantum field theories, Adv.Theor.Math.Phys. 2 (1998) 303-334.
 Dirk Kreimer, On Overlapping Divergences, Commun.Math.Phys. 204 (1999) 669.
- [6] Thomas Krajewski, Raimar Wulkenhaar, On Kreimer’s Hopf algebra structure of Feynman graphs, Eur.Phys.J. C7 (1999) 697-708.
 D.Kreimer, R.Delbourgo, Using the Hopf Algebra Structure of QFT in Calculations, Phys.Rev. D60 (1999) 105025.
 A. Connes, D. Kreimer, Lessons from Quantum Field Theory - Hopf Algebras and Spacetime Geometries, Lett.Math.Phys. 48 (1999) 85-96.
 Alain Connes, Dirk Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem I: the Hopf algebra structure of graphs and the main theorem, Commun.Math.Phys. 210 (2000) 249-273.
- [7] V.Chari and A.Pressley, A guide to quantum groups, CUP (Cambridge:1995), pp. 100–133.
- [8] D.J.Broadhurst, D.Kreimer, Combinatoric explosion of renormalization tamed by Hopf algebra: 30-loop Pade-Borel resummation, Phys.Lett. B475 (2000) 63-70.